Algebraic effects in Montague semantics

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Outline

1. Side effects in linguistic semantics

2. Algebraic effects and handlers

3. Making it Montagovian

4. Quantification and dynamism

We are here

1. Side effects in linguistic semantics
2. Algebraic effects and handlers
3. Making it Montagovian
4. Quantification and dynamism

Semantics

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- characterizing semantic knowledge…
  - …i.e., knowledge of *entailments*? *distributional properties*?
- describing how linguistic structure (i.e., syntax) gives rise to the things being characterized (whatever they are)
- describing how pragmatic stuff (e.g., presupposing something, referring to something, expressing something) should affect the things being characterized

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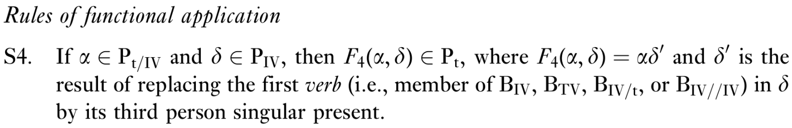
- described how linguistic structure gives rise to meanings, *compositionally*
    - simply typed λ-calculus
- no pragmatic stuff

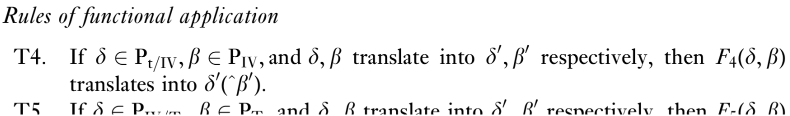
Functional application

Montague 1973:



Rules of functional application

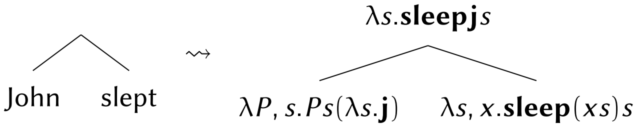
S4. If $\alpha \in \mathrm{P}_{\mathrm{t/IV}}$ and $\delta \in \mathrm{P}_{\mathrm{IV}}$, then $F_4(\alpha, \delta) \in \mathrm{P}_{\mathrm{t}}$, where $F_4(\alpha, \delta) = \alpha\delta'$ and δ' is the result of replacing the first *verb* (i.e., member of B_{IV}, B_{TV}, $\mathrm{B}_{\mathrm{IV/t}}$, or $\mathrm{B}_{\mathrm{IV//IV}}$) in δ by its third person singular present.



Rules of functional application

T4. If $\delta \in \mathrm{P}_{\mathrm{t/IV}}, \beta \in \mathrm{P}_{\mathrm{IV}}$, and δ, β translate into δ', β' respectively, then $F_4(\delta, \beta)$ translates into $\delta'(\hat{}\beta')$.

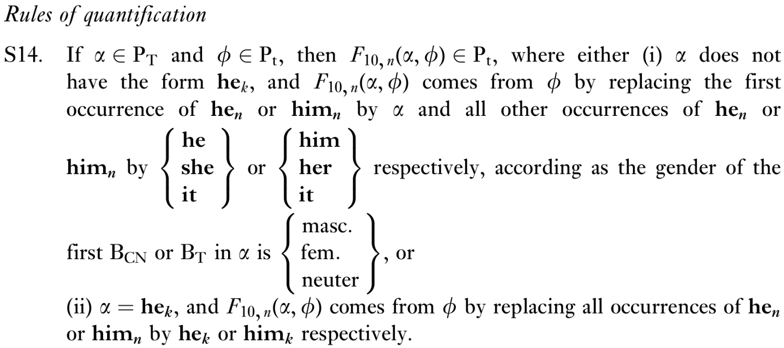
T5. If $\delta \in \mathrm{P}_{\mathrm{IV/T}}$, $\beta \in \mathrm{P}_{\mathrm{T}}$, and δ, β translate into δ', β' respectively, then $F_5(\delta, \beta)$



John slept ⇝ $\lambda s.\mathbf{sleep}\,\mathbf{j}\,s$

$\lambda P, s.Ps(\lambda s.\mathbf{j})$ $\quad$ $\lambda s, x.\mathbf{sleep}(xs)s$

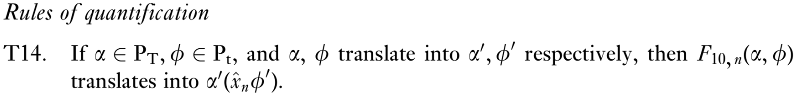
Quantifying in



Rules of quantification

S14. If $\alpha \in \mathrm{P_T}$ and $\phi \in \mathrm{P_t}$, then $F_{10,n}(\alpha, \phi) \in \mathrm{P_t}$, where either (i) α does not have the form $\mathbf{he}_k$, and $F_{10,n}(\alpha, \phi)$ comes from ϕ by replacing the first occurrence of $\mathbf{he}_n$ or $\mathbf{him}_n$ by α and all other occurrences of $\mathbf{he}_n$ or $\mathbf{him}_n$ by $\left\{\begin{matrix}\mathbf{he}\\ \mathbf{she}\\ \mathbf{it}\end{matrix}\right\}$ or $\left\{\begin{matrix}\mathbf{him}\\ \mathbf{her}\\ \mathbf{it}\end{matrix}\right\}$ respectively, according as the gender of the first $\mathrm{B_{CN}}$ or $\mathrm{B_T}$ in α is $\left\{\begin{matrix}\text{masc.}\\ \text{fem.}\\ \text{neuter}\end{matrix}\right\}$, or

(ii) $\alpha = \mathbf{he}_k$, and $F_{10,n}(\alpha, \phi)$ comes from ϕ by replacing all occurrences of $\mathbf{he}_n$ or $\mathbf{him}_n$ by $\mathbf{he}_k$ or $\mathbf{him}_k$ respectively.



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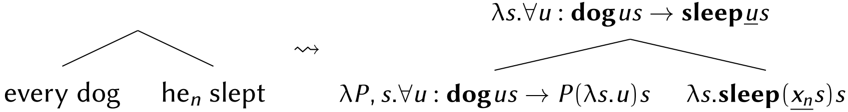
Quantifying in

Every dog slept



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he$_n$ slept

$\rightsquigarrow$

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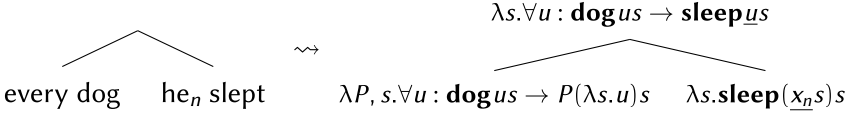

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Not compositional

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    - Monads (Shan, 2002; Charlow, 2014)
    - Idioms (Kobele, 2018)

Side effects

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      ```
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Theories of side effects (e.g., monads) provide interfaces to impure behavior.

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Linguistic side effects

The effectful approach:

- identity linguistic phenomenon that appears to behave “impurely”, i.e., by subverting compositionality
  - e.g., quantification, anaphora, conventional implicature…
- find an effectful interface that appropriately describes its behavior
- add it to your compositional repertoire!

This talk

- present two monadic interfaces to side effects: one for quantification and one anaphora

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- present two monadic interfaces to side effects: one for quantification and one anaphora
    - the Continuation monad and the State monad, respectively
    - analyses inspired by Charlow (2014)
- show how they may and *may not* be combined
- introduce *algebraic effects*

Monads

a functor $\mathcal{M}$, equipped with two operators, $(\cdot)^\eta$ ('return') and $\gg=$ ('bind')

Definition ($\mathcal{M}$)

$$\mathcal{M} : \mathcal{T} \to \mathcal{T}$$

$$(\cdot)^\eta : a \to \mathcal{M}(a)$$

$$(\gg=) : \mathcal{M}(a) \to (a \to \mathcal{M}(b)) \to \mathcal{M}(b)$$

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Intuitively: $\mathcal{M}(a)$ is the space where the side effects of some value of type a happen.

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The operators must satisfy the Monad Laws.

The Monad Laws

$$v^{\eta} \ggg k = kv \qquad \text{(Left Identity)}$$

$$m \ggg \lambda v.v^{\eta} = m \qquad \text{(Right Identity)}$$

$$(m \ggg n) \ggg o = m \ggg \lambda v.nv \ggg o \qquad \text{(Associativity)}$$

First case: quantification

In the Continuation monad, scope-taking is a kind of side effect.

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$$m \gg\!= k = \lambda c.m(\lambda v.kvc)$$

How it works

① Ashley hugged every dog.

How it works

1. Ashley hugged every dog.

$$\begin{aligned} \texttt{ashley} &= \mathbf{a}^{\eta} : \mathcal{C}(e) \\ \texttt{hugged} &= \mathbf{hug}^{\eta} : \mathcal{C}(e \to t) \\ \texttt{every} &= \lambda P, c.\forall x : Px \to cx : (e \to t) \to (e \to t) \to t \\ \texttt{dog} &= \mathbf{dog} : e \to t \end{aligned} \qquad \text{(Lexicon)}$$

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(\triangleright) &: \mathcal{C}(a \to b) \to \mathcal{C}(a) \to \mathcal{C}(b) && \text{(Grammar)}\\
m \triangleright n &= m \ggg \lambda f.n \ggg \lambda x.(f\,x)^{\eta}\\
&= \lambda c.m(\lambda f.n(\lambda x.c(f\,x)))
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expand $\mathtt{every}\mathbf{dog}$...

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to obtain a proposition, apply to the identity function…

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Summary

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Using continuations to manage scope-taking:

- scopal expressions take scope over their continuations, which are reified as they compose
- values take scope trivially (applying Montague's "lift")

Second case: anaphora

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$$\begin{aligned}(\triangleright) &: \mathcal{S}(a \to b) \to \mathcal{S}(a) \to \mathcal{S}(b) && \text{(Grammar)}\\ m \triangleright n &= m \ggg \lambda f.n \ggg \lambda x.(f\,x)^{\eta}\\ &= \lambda s.\mathtt{let}\ \langle f, s'\rangle = ms\ \mathtt{in}\ \mathtt{let}\ \langle x, s''\rangle = ns'\ \mathtt{in}\ \langle f\,x, s''\rangle\end{aligned}$$

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$$\begin{aligned}(\cdot)^{\blacktriangleright} &: \mathcal{S}(e) \to \mathcal{S}(e)\\ m^{\blacktriangleright} &= m \ggg \lambda x, s.\langle x{::}s, x\rangle\end{aligned}$$

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Summary

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- values are trivially stateful, by passing the environment on, untouched

Combining quantification and anaphora

How might one do this?

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How might one do this?

Answer: one may use *monad transformers* (the strategy adopted by Shan (2002), and then, by Charlow (2014)).

Monad transformers

$\mathcal{C}$ and $\mathcal{S}$ are associated with corresponding monad transformers, $\mathcal{C}_T$ and $\mathcal{S}_T$.

Definition ($\mathcal{M}_T$)

$$\begin{aligned}
\mathcal{M}_T &: (\mathcal{T} \rightarrow \mathcal{T}) \rightarrow \mathcal{T} \rightarrow \mathcal{T} \\
(\cdot)^{\eta} &: a \rightarrow \mathcal{M}_T(\mathcal{M}_0)(b) \\
(\ggg) &: \mathcal{M}_T(\mathcal{M}_0)(a) \rightarrow (a \rightarrow \mathcal{M}_T(\mathcal{M}_0)(b)) \rightarrow \mathcal{M}_T(\mathcal{M}_0)(b)
\end{aligned}$$

given one of $\mathcal{C}$ or $\mathcal{S}$ as the *underlying monad*, we may apply one of $\mathcal{S}_T$ or $\mathcal{C}_T$ to it…

The Continuation monad transformer

Definition ($\mathcal{C}_T$)

$$\mathcal{C}_T(\mathcal{M}_0)(a) : (a \to \mathcal{M}_0(o)) \to \mathcal{M}_0(o)$$

$$\begin{aligned}(\cdot)^\eta &: a \to (a \to \mathcal{M}_0(o)) \to \mathcal{M}_0(o) \\ v^\eta &= \lambda c.cv\end{aligned}$$

$$\begin{aligned}(\ggcurly) &: ((a \to \mathcal{M}_0(o)) \to \mathcal{M}_0(o)) \\ &\to (a \to (b \to \mathcal{M}_0(o)) \to \mathcal{M}_0(o)) \\ &\to (b \to \mathcal{M}_0(o)) \to \mathcal{M}_0(o) \\ m \ggcurly k &= \lambda c.m(\lambda v.kvc)\end{aligned}$$

The State monad transformer

Definition ($\mathcal{S}_T$)

$$\mathcal{S}_T(\mathcal{M}_0)(a) : s \to \mathcal{M}_0((s, a))$$

$$\begin{aligned}(\cdot)^\eta &: a \to s \to \mathcal{M}_0((s, a))\\ v^\eta &= \lambda s.\langle s, v\rangle^\eta\end{aligned}$$

$$\begin{aligned}(\ggg\!=) : &\ (s \to \mathcal{M}_0((s, a)))\\ &\to (a \to (s \to \mathcal{M}_0((s, b))))\\ &\to s \to \mathcal{M}_0((s, b))\end{aligned}$$

$$m \ggg\!= k = \lambda s.ms \ggg\!= \lambda p.\mathtt{let}\ \langle s', v\rangle = p\ \mathtt{in}\ kvs'$$

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This general strategy can be made to work extremely well (Charlow, 2014).

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  - but a generic meaning for *every* cannot be written…we are required to know what $\mathcal{M}_0$ is!
  - even then, the meaning the quantifier will be somewhat stipulative, e.g., to account for the data above (though, it can be made to follow from a small set of primitives, as in Charlow (2014))

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The transformers approach, when used generically, prevents us from writing meanings. When used non-generically, it loses extensibility.

Might we salvage our individual analyses in some other way? In doing so, might we account for data like (1)?

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1 Side effects in linguistic semantics

2 Algebraic effects and handlers

3 Making it Montagovian

4 Quantification and dynamism

Background

Algebraic effects and handlers provide a means of writing extensible code, recently especially popular in functional programming.[1]

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- develops a typed extension of λ-calculus
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- anaphora is approached using the compositional DRT of de Groote (2006)

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- staying in STLC (with unit)
- characterizing anaphora in purely algebraic terms
- sticking with a traditional analysis of quantifiers, i.e., whereon they denote sets of sets

Algebraic signatures

An algebraic signature is a set E of operations, each one associated with a *parameter* p and an *arity* a (both types), along with a special operation η ('return').

$$E = \{\mathrm{op}_{1_{p_1 \rightsquigarrow a_1}}, \ldots, \mathrm{op}_{n_{p_n \rightsquigarrow a_n}}, \eta\}$$

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To say operator $\mathrm{op}_{p \rightsquigarrow a}$ is in signature E means that it has the following type signature:

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η always has the following type signature:

$$\eta : v \to \mathcal{F}_E(v)$$

Algebraic laws

In addition to the signature, an algebra determines a set of equations that must hold among its elements, of the form

$$\mathrm{op}_i(p_i; \ldots) = \mathrm{op}_j(p_j; \ldots)$$

The State algebra (signature)

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The State algebra (laws)

Reading the environment twice is no better than reading it once:

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Putting twice overwrites:

$$\mathtt{put}_{s\rightsquigarrow\star}(g; \lambda\star.\mathtt{put}_{s\rightsquigarrow\star}(g'; k)) = \mathtt{put}_{s\rightsquigarrow\star}(g'; k)$$

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Some example elements of the Quantifier algebra…

- $\mathtt{scope}_{(e\to t)\to t\rightsquigarrow e}(\mathtt{every}\,\mathbf{dog}; \lambda y.\eta(\mathbf{sleep}\,y)) : \mathcal{F}_{\{\mathtt{scope}_{(e\to t)\to t\rightsquigarrow e}\}}(t)$
- $\mathtt{scope}_{(e\to t)\to t\rightsquigarrow e}(\mathtt{every}\,\mathbf{dog}; \lambda y.\mathtt{scope}_{(e\to t)\to t\rightsquigarrow e}(\mathtt{every}\,\mathbf{cat}; \lambda z.\eta(\mathbf{chase}\,z\,y))) : \mathcal{F}_{\{\mathtt{scope}_{(e\to t)\to t\rightsquigarrow e}\}}(t)$

The Quantifier algebra (laws)

Quantifying in:

$$\mathrm{scope}_{(e\to t)\to t\rightsquigarrow e}(q;\lambda x.\eta(kx)) = \eta(qk)$$

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Commuting $\mathtt{scope}_{(e\to t)\to e\rightsquigarrow e}$ past $\mathtt{get}_{\star\rightsquigarrow s}$ and $\mathtt{put}_{s\rightsquigarrow\star}$:

$$\mathtt{scope}_{(e\to t)\to e\rightsquigarrow e}(q; \lambda x.\mathtt{get}_{\star\rightsquigarrow s}(\star; \lambda s.\mathtt{put}_{s\rightsquigarrow\star}(s'; \lambda\star.kxss')))$$

$$= \mathtt{get}_{\star\rightsquigarrow s}(\star; \lambda s.\mathtt{put}_{s\rightsquigarrow\star}(s; \lambda\star.\mathtt{scope}_{(e\to t)\to e\rightsquigarrow e}(q; \lambda x.kxss')))$$

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How to do it

What we want is an encoding of the operations, as well as a way of *translating* λ-terms with lots of operations into ones with fewer operations in a way that respects the algebraic laws.

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In the combined State/Quantifier algebra, the normal form for any element is determined by the laws to be

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for some $f : s \rightarrow s$ and $g : s \rightarrow v$.

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Pairs of such functions f and g can be represented as $\lambda s.\langle f\,s, gs\rangle$ …they are State monadic!

Encoding elements

To encode elements of an algebra, we define a family of functors $\mathcal{F} : \mathcal{T}^*_{\leadsto} \to \mathcal{T} \to \mathcal{T}$, where

- $\mathcal{T}^*_{\leadsto}$ is the free monoid (i.e., of lists) over $\mathcal{T}_{\leadsto} = \{p \leadsto a \mid p, a \in \mathcal{T}\}$

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$$\begin{aligned}\mathcal{F}_{\epsilon}(v) &= v \\ \mathcal{F}_{p \rightsquigarrow a, l}(v) &= (p \to (a \to \mathcal{F}_l(v)) \to o) \to o\end{aligned}$$

Encoding elements

To encode elements of an algebra, we define a family of functors $\mathcal{F} : \mathcal{T}^*_{\rightsquigarrow} \to \mathcal{T} \to \mathcal{T}$, where

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Operations construct “pairs”; returning just returns…

An element of the State/Quantifier algebra

1. Every dog hugged itself.

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$$\begin{array}{l}\mathtt{scope}_{(e\to t)\to t\rightsquigarrow e}(\mathtt{every}\,\mathbf{dog};\\ \qquad\lambda x.\mathtt{get}_{\star\rightsquigarrow s}(\star;\\ \qquad\qquad\lambda s.\mathtt{put}_{s\rightsquigarrow\star}(x{::}s;\\ \qquad\qquad\qquad\lambda\star.\mathtt{get}_{\star\rightsquigarrow s}(\star;\lambda s'.\eta(\mathbf{hug}(\mathtt{sel}\,s')x)))))\end{array}$$

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This will be an expression of type

$$\begin{aligned}&\mathcal{F}_{(e\to t)\to t\rightsquigarrow e,\star\rightsquigarrow s,s\rightsquigarrow\star,\star\rightsquigarrow s}\\ = \;&(((e\to t)\to t)\to(e\to((\star\to(s\to\ldots)\to o')\to o'))\to o)\to o\end{aligned}$$

Handling operations

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We need a family of functions

$$\texttt{handleSentence}_l : \mathcal{F}_l(t) \to \mathcal{F}_{\star \to s, s, \to \star}(t)$$

where $l \in \{(e \to t) \to t \rightsquigarrow e,\ \ \star \rightsquigarrow s,\ \ s \rightsquigarrow \star\}^*$.

We are here

1. Side effects in linguistic semantics

2. Algebraic effects and handlers

3. Making it Montagovian

4. Quantification and dynamism

Predictions

We would like to explain contrasts such as

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In sum

Our algebraic laws predict the contrasts! Crucial is the law that commutes $\mathtt{scope}_{(e \to t) \to t \rightsquigarrow e}$ past $\mathtt{get}_{\star \rightsquigarrow s}$ and $\mathtt{put}_{\star \rightsquigarrow s}$.

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$$= \mathtt{get}_{\star\rightsquigarrow s}(\star; \lambda s.\mathtt{put}_{s\rightsquigarrow\star}(s; \lambda\star.\mathtt{scope}_{(e\to t)\to e\rightsquigarrow e}(q; \lambda x.kxss')))$$

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This law destroys a quantifier’s dynamic potential, rendering it externally static.

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This gives us a new and precise way of characterizing certain old semantic problems about quantification and dynamism:

- when combining algebras, where do any new laws come from? can they come for free?

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